

REGULARIZATION STRATEGY FOR INVERSE PROBLEM FOR 1+1 DIMENSIONAL WAVE EQUATION

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ABSTRACT. An inverse boundary value problem for a 1+1 dimensional wave equation with wave speed $c(x)$ is considered. We give a regularisation strategy for inverting the map $\mathcal{A} : c \mapsto \Lambda$, where Λ is the hyperbolic Neumann-to-Dirichlet map corresponding to the wave speed c . More precisely, we consider the case when we are given a perturbation of the Neumann-to-Dirichlet map $\tilde{\Lambda} = \Lambda + \mathcal{E}$, where \mathcal{E} corresponds to the measurement errors, and reconstruct an approximate wave speed \tilde{c} . We emphasize that $\tilde{\Lambda}$ may not be in the range of the map \mathcal{A} . We show that the reconstructed wave speed \tilde{c} satisfies $\|\tilde{c} - c\|_{L^\infty} < C\|E\|^{1/18}$. Our regularization strategy is based on a new formula to compute c from Λ .

Keywords: Inverse problem, regularization theory, wave equation.

1. INTRODUCTION

We consider an inverse boundary value problem for the wave equation

$$(\partial_t^2 - c(x)^2 \partial_x^2)u(t, x) = 0.$$

We introduce a regularization strategy to recover the sound speed $c(x)$ by using the knowledge of perturbed Neumann-to-Dirichlet map $\tilde{\Lambda}$. Our approach is based on the Boundary Control method [2, 6, 53].

A variant of the Boundary Control method, called the iterative time-reversal control method, was introduced in [9]. The method was later modified in [15] to focus the energy of a wave at a fixed time and in [46] to solve an inverse obstacle problem for the wave equation. Here we introduce yet another modification of the iterative time-reversal control method that is tailored for the 1+1 dimensional wave equation.

Classical regularization theory is explained in [16]. Iterative regularization of both linear and nonlinear inverse problems and convergence rates are discussed in Hilbert space setting in [10, 18, 20, 41, 43] and in Banach space setting in [19, 23, 24, 30, 47, 48, 49]. Our new results

give a direct regularization method for the nonlinear inverse problem for the wave equation. The result contains an explicit (but not necessarily optimal) convergence rate.

By direct methods for non-linear problems we mean explicit construction of a non-linear map solving the problem without resorting to a local optimisation method. In our case the map is given by (55). The advantage of direct approaches is that they do not suffer from the possibility that the algorithm converges to a local minimum. In particular, they do not require a priori knowledge that the solution is in a small neighbourhood of a given function. There are currently only few regularized direct methods for non-linear inverse problems. An example is a regularisation algorithm for the inverse problem for the conductivity equation in [31]. Also, a direct regularized inversion for blind deconvolution is presented in [21].

1.1. Statement of the results. Let $C_0, C_1, L, m > 0$ and define the space of velocity functions

$$(1) \quad \mathcal{D}(\mathcal{A}) := \{c \in L^\infty(M); C_0 \leq c(x) \leq C_1, \\ \|c\|_{C^2(M)} \leq m, c - 1 \in C_0^2(0, L)\},$$

where we denote by M the half axis $[0, \infty)$. Let

$$(2) \quad T > \frac{L}{C_0}.$$

For $c \in \mathcal{D}(\mathcal{A})$ and $f \in L^2(0, 2T)$, the boundary value problem

$$(3) \quad \begin{aligned} (\partial_t^2 - c(x)^2 \partial_x^2)u(t, x) &= 0 \quad \text{in } (0, 2T) \times M, \\ \partial_x u(t, 0) &= f(t), \\ u|_{t=0} = \partial_t u|_{t=0} &= 0, \end{aligned}$$

has a unique solution $u^f \in H^1((0, 2T) \times M)$. Using this solution we define the Neumann-to-Dirichlet operator

$$(4) \quad \Lambda : L^2(0, 2T) \rightarrow L^2(0, 2T), \quad \Lambda f = u^f|_{x=0}.$$

We define for a Banach space E

$$\mathcal{L}(E) := \{A : E \rightarrow E; A \text{ is linear and continuous}\}.$$

Let $X = L^\infty(M)$ and $Y = \mathcal{L}(L^2(0, 2T))$. We define the direct map

$$(5) \quad \mathcal{A} : \mathcal{D}(\mathcal{A}) \subset X \rightarrow Y, \quad \mathcal{A}(c) = \Lambda.$$

We show in Appendix A, Theorem 4, that the maps (4) and (5) are continuous. Here the range $\text{Ran}(\mathcal{A}) = \mathcal{A}(\mathcal{D}(\mathcal{A}))$ and the domain $\mathcal{D}(\mathcal{A})$ are equipped with the topologies of Y and X , respectively.

We consider the inverse problem to recover the velocity function c by using the boundary measurements Λ . It is well-known that \mathcal{A} is invertible. Let us record the following:

Theorem 1. *The inverse map*

$$\mathcal{A}^{-1} : \text{Ran}(\mathcal{A}) \subset Y \rightarrow \mathcal{D}(\mathcal{A}) \subset X, \quad \mathcal{A}^{-1}(\Lambda) = c,$$

is continuous.

For the convenience of the reader we give a proof of Theorem 1 in Section 2, where we also give a new formula to compute c from Λ . Our main result concerns perturbations of the Neumann-to-Dirichlet operator of the form

$$(6) \quad \tilde{\Lambda} = \Lambda + \mathcal{E},$$

where $\mathcal{E} \in Y$ models the measurement error. We assume that $\|\mathcal{E}\|_Y \leq \epsilon$, where $\epsilon > 0$ is known. In this situation we can not use the map \mathcal{A}^{-1} to calculate function c since $\tilde{\Lambda}$ may not be in the range $\mathcal{R}(\mathcal{A})$. We recall the definition of a regularization strategy, see e.g. [16] and [30].

Definition 1. *Let X, Y be Banach spaces and $\mathcal{D}(\mathcal{A}) \subset X$. Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow Y$ be continuous mapping. Let $\alpha_0 \in (0, \infty]$. A family of continuous maps $\mathcal{R}_\alpha : Y \rightarrow X$ parametrized by $0 < \alpha < \alpha_0$ is called a regularization strategy if*

$$\lim_{\alpha \rightarrow 0} \mathcal{R}_\alpha(\mathcal{A}(c)) = c$$

for every $c \in \mathcal{D}(\mathcal{A})$. A regularization strategy is called admissible, if the parameter α is chosen as a function of the noise level ϵ so that $\lim_{\epsilon \rightarrow 0} \alpha(\epsilon) = 0$ and for every $c \in \mathcal{D}(\mathcal{A})$

$$\lim_{\epsilon \rightarrow 0} \sup \left\{ \left\| \mathcal{R}_{\alpha(\epsilon)} \tilde{\Lambda} - c \right\|_X : \tilde{\Lambda} \in Y, \left\| \tilde{\Lambda} - \mathcal{A}(c) \right\|_Y \leq \epsilon \right\} = 0.$$

Figure 1 gives us a schematic illustration of regularization.

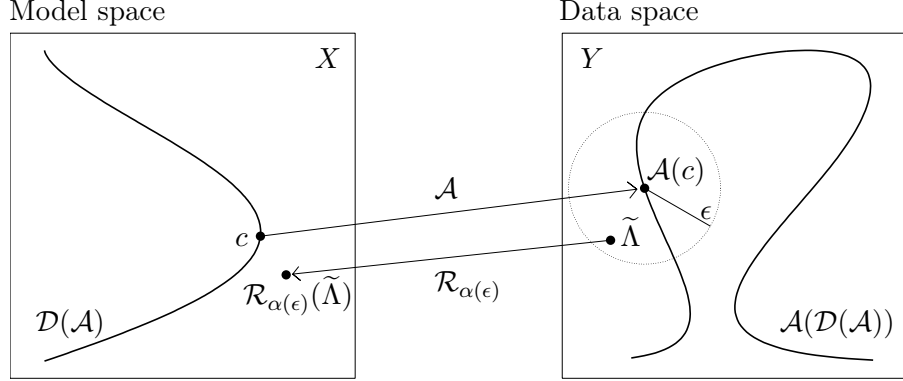


FIGURE 1. The idea of regularization is to construct a family $\mathcal{R}_{\alpha(\epsilon)}$ of continuous maps from the data space Y to the model space X in such a way that c can be approximately recovered from noisy data $\tilde{\Lambda}$. For a smaller noise level ϵ the approximation $\mathcal{R}_{\alpha(\epsilon)}(\tilde{\Lambda})$ is closer to c . More details and a similar figure in a more general setting can be found in [44, Fig. 11.5].

The main result is presented next and says that we have an admissible regularization strategy inverting \mathcal{A} .

Theorem 2. *There exists an admissible regularization strategy \mathcal{R}_α with the choice of parameter*

$$\alpha(\epsilon) = 2^{\frac{13}{9}} T^{\frac{4}{9}} \epsilon^{\frac{4}{9}}$$

that satisfies the following: For every $c \in \mathcal{D}(\mathcal{A})$ there is $\epsilon_0, C > 0$ such that

$$\sup \left\{ \left\| \mathcal{R}_{\alpha(\epsilon)} \tilde{\Lambda} - c \right\|_X : \tilde{\Lambda} \in Y, \left\| \mathcal{A}(c) - \tilde{\Lambda} \right\|_Y \leq \epsilon \right\} \leq C \epsilon^{\frac{1}{18}},$$

for all $\epsilon \in (0, \epsilon_0)$.

We will give explicit choices of \mathcal{R}_α and ϵ_0 , see (54) and (55) below.

1.2. Previous literature. From the point of view of uniqueness questions, the inverse problem for the 1+1 dimensional wave equation is equivalent with the one dimensional inverse boundary spectral problem. The latter problem was thoroughly studied in 1950s [17, 32, 42] and we refer to [22, pp. 65-67] for a historical overview. In 1960s Blagoveščenskii [12, 13] developed an approach to solve the inverse problem for the 1+1 dimensional wave equation without reducing the problem to the inverse boundary spectral problem. This and later dynamical methods

have the advantage over spectral methods that they require data only on a finite time interval.

The method in the present paper is a variant of the Boundary Control method that was pioneered by M. Belishev [2] and developed by M. Belishev and Y. Kurylev [5, 6] in late 80s and early 90s. Of crucial importance for the method was the result of D. Tataru [53] concerning a Holmgren-type uniqueness theorem for non-analytic coefficients. The Boundary Control method for multidimensional inverse problems has been summarized in [3, 26], and considered for 1+1 dimensional scalar problems in [4, 7] and for multidimensional scalar problems in [25, 28, 33, 38, 39]. For systems it has been considered in [34, 35]. Stability results for the method have been considered in [1] and [29].

The inverse problem for the wave equation can be solved also by using complex geometrical optics solutions. These solutions were developed in the context of elliptic inverse boundary value problems [52], and in [45] they were employed to solve an inverse boundary spectral problem. Local stability results can be proven using (real) geometrical optics solutions [8, 50, 51], and in [40] a local stability result was proved by using ideas from the Boundary Control method together with complex geometrical optics solutions. Finally we mention the important method based on Carleman estimates [14] that can be used to show stability results when the initial data for the wave equation is non-vanishing.

2. MODIFICATION OF THE ITERATIVE TIME-REVERSAL CONTROL METHOD

In this section we prove Theorem 1 in such a way that we can utilize the proof to construct a regularization strategy as in Theorem 2. Let Λ be as defined in (4). Let $r \in [0, T]$. We define linear operators in Y by

$$(7) \quad \begin{aligned} Jf(t) &= \frac{1}{2} \int_0^{2T} 1_\Delta(t, s) f(s) ds, \\ Rf(t) &= f(2T - t), \quad K = J\Lambda - R\Lambda R J, \\ Bf(t) &= 1_{(0, T)}(t) \int_t^T f(s) ds, \quad P_r f(t) = 1_{(0, r)}(t) f(t), \end{aligned}$$

where

$$1_\Delta(t, s) = \begin{cases} 1, & t + s \leq 2T \text{ and } s > t > 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$1_{(0,r)}(t) = \begin{cases} 1, & t \in (0, r), \\ 0, & \text{otherwise.} \end{cases}$$

Let $f \in L^2(0, 2T)$. Using solution $u^f \in H^1((0, 2T) \times M)$ of (3) we define

$$(8) \quad U_T : L^2(0, 2T) \mapsto H^1(M), \quad U_T f = u^f(T).$$

Let us denote $dV = c^{-2}dx$ and recall the Blagovestchenskii identities

$$(9) \quad \begin{aligned} \langle u^f(T), 1 \rangle_{L^2(M; dV)} &= \langle f, B1 \rangle_{L^2(0, 2T)}, \\ \langle u^f(T), u^h(T) \rangle_{L^2(M; dV)} &= \langle f, Kh \rangle_{L^2(0, 2T)}. \end{aligned}$$

The identities (9) originate from the work by Blagovestchenskii [11] and their proofs can be found e.g. in [9]. We define the domain of influence

$$(10) \quad M(r) = \{x \in M; d(x, 0) \leq r\},$$

where $d(x, 0) = \int_0^x \frac{1}{c(t)} dt$. We use the following result that is closely related to [9]

Theorem 3. *Let $r \in [0, T]$ and $\alpha > 0$. Let K, B , and P_r be as defined in (7). Let us define*

$$(11) \quad S_r = \{f \in L^2(0, 2T) : \text{supp}(f) \subset [T - r, T]\}.$$

Then the regularized minimization problem

$$\min_{f \in S_r} \left(\langle f, Kf \rangle_{L^2(0, 2T)} - 2\langle f, B1 \rangle_{L^2(0, 2T)} + \alpha \|f\|_{L^2(0, 2T)}^2 \right),$$

has unique minimizer

$$(12) \quad f_{\alpha, r} = (P_r K P_r + \alpha)^{-1} P_r B1$$

and the map $r \mapsto f_{\alpha, r}$ is continuous $[0, T] \rightarrow L^2(0, 2T)$. Moreover $u^{f_{\alpha, r}}(T)$ converges to the indicator function of the domain of influence,

$$(13) \quad \lim_{\alpha \rightarrow 0} \|u^{f_{\alpha, r}}(T) - 1_{M(r)}\|_{L^2(M; dV)} = 0.$$

For the convenience of the reader we give a proof.

Proof of Theorem 3. Let $\alpha > 0$ and let $f \in S_r$. We define the energy function

$$(14) \quad E(f) := \langle f, Kf \rangle_{L^2(0, 2T)} - 2\langle f, B1 \rangle_{L^2(0, 2T)} + \alpha \|f\|_{L^2(0, 2T)}^2.$$

The finite speed of wave propagation implies $\text{supp}(u^f(T)) \subset M(r)$. Using (9) we can write

(15)

$$E(f) = \|u^f(T) - 1_{M(r)}\|_{L^2(M;dV)}^2 - \|1_{M(r)}\|_{L^2(M;dV)}^2 + \alpha \|f\|_{L^2(0,2T)}^2.$$

Let $(f_j)_{j=1}^\infty \subset S_r$ be such that

$$\lim_{j \rightarrow \infty} E(f_j) = \inf_{f \in S_r} E(f).$$

Then

$$\alpha \|f_j\|_{L^2(0,2T)}^2 \leq E(f_j) + \|1_{M(r)}\|_{L^2(M;dV)}^2,$$

and we see that $(f_j)_{j=1}^\infty$ is bounded in S_r . As S_r is a Hilbert space, there is a subsequence of $(f_j)_{j=1}^\infty$ converging weakly in S_r . Let us denote the limit by $f_\infty \in S_r$ and the subsequence still by $(f_j)_{j=1}^\infty$.

By Theorem 4 in Appendix A below, the map $U_T : f \mapsto u^f(T)$ is bounded

$$U_T : L^2(0, 2T) \rightarrow H^1(M).$$

The embedding $I : H^1(M) \hookrightarrow L^2(M)$ is compact and thus $U_T : f \mapsto u^f(T)$ is a compact operator

$$U_T : L^2(0, 2T) \rightarrow L^2(M).$$

Hence we have a subsequence $(f_j)_{j=1}^\infty$ for which $u^{f_j}(T) \rightarrow u^{f_\infty}(T)$ in $L^2(M)$ as $j \rightarrow \infty$. Moreover, the weak convergence implies

$$\|f_\infty\|_{L^2(0,2T)} \leq \liminf_{j \rightarrow \infty} \|f_j\|_{L^2(0,2T)}.$$

Hence

$$\begin{aligned} E(f_\infty) &= \lim_{j \rightarrow \infty} \|u^{f_j}(T) - 1_{M(r)}\|_{L^2(M;dV)}^2 - \|1_{M(r)}\|_{L^2(M;dV)}^2 + \alpha \|f_\infty\|_{L^2(0,2T)}^2 \\ &\leq \lim_{j \rightarrow \infty} \|u^{f_j}(T) - 1_{M(r)}\|_{L^2(M;dV)}^2 - \|1_{M(r)}\|_{L^2(M;dV)}^2 + \alpha \liminf_{j \rightarrow \infty} \|f_j\|_{L^2(0,2T)}^2 \\ &= \liminf_{j \rightarrow \infty} E(f_j) = \inf_{f \in S_r} E(f), \end{aligned}$$

and thus $f_\infty \in S_r$ is a minimizer for (14). We denote by D_h the Fréchet derivative to direction h . If

$$0 = D_h E(f) = 2\langle h, P_r K P_r f \rangle_{L^2(0,2T)} - 2\langle h, P_r B 1 \rangle_{L^2(0,2T)} + 2\alpha \langle h, f \rangle_{L^2(0,2T)},$$

for all $h \in L^2(0, 2T)$, then

$$(P_r K P_r + \alpha)f = P_r B 1.$$

Using (9) we have

$$\langle (P_r K P_r + \alpha)f, f \rangle_{L^2(0,2T)} = \langle u^{P_r f}(T), u^{P_r f}(T) \rangle_{L^2(M;dV)} + \langle \alpha f, f \rangle_{L^2(0,2T)}.$$

Operator $P_r K P_r + \alpha$ is coersive when $\alpha > 0$. The Lax-Milgram Theorem implies that it is invertible, and we have an expression for minimizer

$$f_{\alpha,r} := f_\infty = (P_r K P_r + \alpha)^{-1} P_r B 1.$$

According to [53], see also [27], we know that

$$\{u^f(T) \in L^2(M(r)); f \in S_r\}$$

is dense in $L^2(M(r))$. Let $\epsilon > 0$ and fix $f_\epsilon \in S_r$ such that

$$(16) \quad \|u^{f_\epsilon}(T) - 1_{M(r)}\|_{L^2(M;dV)}^2 \leq \epsilon.$$

Using (15) we have

$$\|u^{f_{\alpha,r}}(T) - 1_{M(r)}\|_{L^2(M;dV)}^2 \leq E(f_{\alpha,r}) + \|1_{M(r)}\|_{L^2(M;dV)}^2.$$

Because $E(f_{\alpha,r}) \leq E(f_\epsilon)$ we have

$$\begin{aligned} \|u^{f_{\alpha,r}}(T) - 1_{M(r)}\|_{L^2(M;dV)}^2 &\leq \|u^{f_\epsilon}(T) - 1_{M(r)}\|_{L^2(M;dV)}^2 + \alpha \|f_\epsilon\|^2. \\ &\leq \epsilon + \alpha \|f_\epsilon\|^2. \end{aligned}$$

Using (16) we may choose first small $\epsilon > 0$ and then small $\alpha > 0$ to see that $u^{f_\alpha}(T)$ tends to $1_{M(r)}$ in $L^2(M)$ as $\alpha \rightarrow 0$. \square

See Figure 2 for a visualization of $M(r)$.

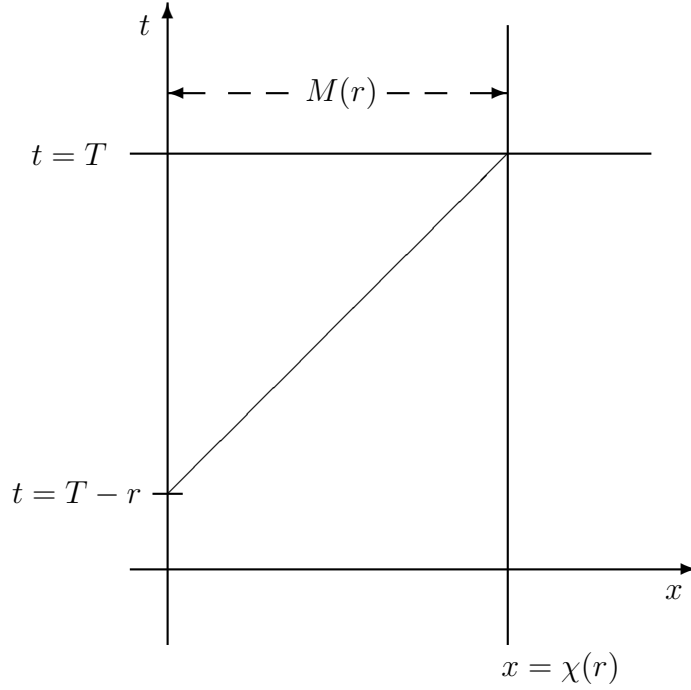


FIGURE 2. When sending information from an interval of length r , that is when $\text{supp}(f) \subset [T-r, T]$, the solution $u^f(x, T)$ is supported in the domain of influence $M(r)$.

We define the travel time coordinates for $x \in M$ by

$$\tau : [0, \infty) \rightarrow [0, \infty), \quad \tau(x) = d(x, 0).$$

The function τ is strictly increasing and we denote its inverse by

$$\chi = \tau^{-1} : [0, \infty) \rightarrow [0, \infty).$$

We have

$$(17) \quad \chi(0) = 0, \quad \chi'(t) = \frac{1}{\tau'(\chi(t))} = c(\chi(t)).$$

Thus denoting $v(t) = c(\chi(t))$ and use $V(r)$ to denote the volume of $M(r)$ with respect to the measure dV we have

$$(18) \quad V(r) = \|1_{M(r)}\|_{L^2(M; dV)}^2 = \int_0^{\chi(r)} \frac{dx}{c(x)^2} = \int_0^r \frac{\chi'(t) dt}{v(t)^2} = \int_0^r \frac{dt}{v(t)}.$$

Note that $M(r) = [0, \chi(r)]$. In particular, $V(r)$ determines the wave speed in the travel time coordinates,

$$(19) \quad v(r) = \frac{1}{\partial_r V(r)},$$

and also in the original coordinates since

$$(20) \quad c(x) = v(\chi^{-1}(x)), \quad \chi(t) = \int_0^t v(t') dt'.$$

Using Theorem 3 and (9) we have a method to compute the volumes of the domains of influence

$$(21) \quad V(r) = \|1_{M(r)}\|_{L^2(M; dV)}^2 = \lim_{\alpha \rightarrow 0} \langle f_{\alpha, r}, B1 \rangle_{L^2(0, 2T)},$$

where $r \in [0, T]$. We are ready to prove Theorem 1.

Proof of Theorem 1. For a given measurement Λ , Theorem 3 and equations (19), (20), (21) give us a way to calculate for all $x \in (0, L)$ the value of the velocity function

$$c(x) = v(\chi^{-1}(x)) = \mathcal{A}^{-1}(\Lambda)(x).$$

As we assumed that outside of the interval $(0, L)$ the function c is identically one, the proof for the existence of inverse map \mathcal{A}^{-1} is complete. Since the direct map \mathcal{A} is continuous and $\mathcal{D}(\mathcal{A})$ is relatively compact in X , we see that \mathcal{A}^{-1} is a continuous map. \square

3. STABILITY OF REGULARIZED PROBLEM

In this section we prove Theorem 2. We will construct the operator $\mathcal{R}_{\alpha(\epsilon)}$ as a composition of several operators. The construction is motivated by the proof of Theorem 1. We define for a Banach space E

$$\mathcal{K}(E) = \{A \in \mathcal{L}(E); A \text{ is compact}\}.$$

Let J, R be as defined in (7). Using (7) we see that $J \in \mathcal{K}(L^2(0, 2T))$. We define

$$(22) \quad \begin{aligned} \mathbf{K} : Y &\rightarrow \mathcal{K}(L^2(0, 2T)), & \mathbf{K}\tilde{\Lambda} &= J\tilde{\Lambda} - R\tilde{\Lambda}RJ. \\ \mathbf{H} : Y &\mapsto C([0, T], Y), & \mathbf{H}\tilde{\Lambda} &= r \mapsto P_r(\mathbf{K}\tilde{\Lambda})P_r. \end{aligned}$$

Proposition 1. *We have $\|\mathbf{H}\|_{Y \rightarrow C([0, T], Y)} \leq 2T$.*

Proof. Let $r \in [0, T]$. We have estimates $\|P_r\|_Y \leq 1$, $\|R\|_Y \leq 1$, $\|J\|_Y \leq T$, and

$$\|\mathbf{H}\tilde{\Lambda}(r)\|_{\mathcal{L}(L^2(0, 2T))} \leq 2\|J\|_{\mathcal{L}(L^2(0, 2T))} \|\tilde{\Lambda}\|_{\mathcal{L}(L^2(0, 2T))} \leq 2T \|\tilde{\Lambda}\|_{\mathcal{L}(L^2(0, 2T))}.$$

Thus

$$\|\mathbf{H}\|_{Y \rightarrow L^\infty([0, T], Y)} \leq 2T.$$

It remains to show that $r \mapsto \mathbf{H}\tilde{\Lambda}(r)$ is continuous. Let us denote $\tilde{K} = \mathbf{K}\tilde{\Lambda}$. Let $r, s \in [0, T]$. We use the singular value decomposition

for the compact operator \tilde{K} . There are orthonormal bases $\{\phi_n\}_{n=1}^\infty \in L^2(0, 2T)$ and $\{\psi_n\}_{n=1}^\infty \in L^2(0, 2T)$ such that

$$(23) \quad \tilde{K}f = \sum_{n=1}^{\infty} \mu_n \langle f, \phi_n \rangle_{L^2(0, 2T)} \psi_n,$$

for all $f \in L^2(0, 2T)$, where $\mu_n \in \mathbb{R}$ are the singular values of \tilde{K} . We define the family $\{\tilde{K}^m\}_{m=1}^\infty$ of finite rank operators by the formula

$$(24) \quad \tilde{K}^m f = \sum_{n=1}^m \mu_n \langle f, \phi_n \rangle_{L^2(0, 2T)} \psi_n.$$

Then

$$(25) \quad \begin{aligned} & \left\| P_r \tilde{K} P_r f - P_s \tilde{K} P_s f \right\|_{L^2(0, 2T)} \\ & \leq \left\| P_r \tilde{K} P_r f - P_r \tilde{K}^m P_r f \right\|_{L^2(0, 2T)} + \left\| P_r \tilde{K}^m P_r f - P_s \tilde{K}^m P_r f \right\|_{L^2(0, 2T)} + \\ & \quad \left\| P_s \tilde{K}^m P_r f - P_s \tilde{K}^m P_s f \right\|_{L^2(0, 2T)} + \left\| P_s \tilde{K}^m P_s f - P_s \tilde{K} P_s f \right\|_{L^2(0, 2T)}. \end{aligned}$$

Let $\epsilon > 0$ and let $\|f\|_{L^2(0, 2T)} \leq 1$. By choosing m large enough we have

$$\left\| P_r \tilde{K} P_r f - P_r \tilde{K}^m P_r f \right\|_{L^2(0, 2T)} + \left\| P_s \tilde{K}^m P_s f - P_s \tilde{K} P_s f \right\|_{L^2(0, 2T)} \leq \frac{\epsilon}{2}.$$

Applying projections to (24) we see that

$$P_s \tilde{K}^m P_r f = \sum_{n=1}^m \mu_n \langle f, P_r \phi_n \rangle P_s \psi_n.$$

For the second term in the sum (25) we have an estimate

$$\begin{aligned} & \left\| P_r \tilde{K}^m P_r f - P_s \tilde{K}^m P_r f \right\|_{L^2(0, 2T)} = \left\| \sum_{n=1}^m \mu_n \langle f, P_r \phi_n \rangle (P_r - P_s) \psi_n \right\|_{L^2(0, 2T)} \\ & \leq \sum_{n=1}^m |\mu_n| \|(P_r - P_s) \phi_n\|_{L^2(0, 2T)} \leq C(m) |r - s|^{\frac{1}{2}}. \end{aligned}$$

For the third term in the sum we have an analogous estimate

$$\left\| P_s \tilde{K}^m P_r f - P_s \tilde{K}^m P_s f \right\|_{L^2(0, 2T)} \leq C(m) |r - s|^{\frac{1}{2}}.$$

Putting these estimates together and choosing $|r - s| \leq \delta(\epsilon) = \frac{\epsilon^2}{4C(m)^2}$, we see that

$$\left\| P_r \tilde{K} P_r - P_s \tilde{K} P_s \right\|_Y \leq \epsilon. \quad \square$$

Let us define

$$(26) \quad M_1 = \sup\{\|\mathcal{A}(c)\|_{\mathcal{L}(L^2(0,2T))}; c \in \mathcal{D}(\mathcal{A})\}.$$

Using the continuity of \mathcal{A} , see Theorem 4 below, we see that $M_1 < \infty$. We define $M_2 = 2TM_1$. Let $c \in \mathcal{D}(\mathcal{A})$ and denote $\Lambda = \mathcal{A}(c)$. We use again the notations $H = \mathbf{H}\Lambda$, $\tilde{H} = \mathbf{H}\tilde{\Lambda}$ and $\tilde{H}_r = \mathbf{H}\tilde{\Lambda}(r)$. Using Proposition 1 we have

$$(27) \quad \|H\|_{C([0,T],Y)} \leq M_2.$$

We define $M_3 = M_2 + 3$ and a family $\{\Psi_\alpha^Z\}_{\alpha \in (0,2]} \in C(\mathbb{R})$ by

$$\Psi_\alpha^Z(s) = \begin{cases} 0, & \text{if } s > M_3 - \frac{\alpha}{4}, \\ -\frac{4}{\alpha}s + \frac{4M_3}{\alpha} - 1, & \text{if } s \in (M_3 - \frac{\alpha}{2}, M_3 - \frac{\alpha}{4}], \\ 1, & \text{if } s \leq M_3 - \frac{\alpha}{2}. \end{cases}$$

For $\alpha \in (0, 2]$ we define

$$(28) \quad \mathbf{Z}_\alpha : C([0, T], Y) \rightarrow C([0, T], Y),$$

$$\mathbf{Z}_\alpha(\tilde{H}) = r \mapsto \Psi_\alpha^Z\left(\left\|M_3 - (\tilde{H} + \alpha)\right\|_{C([0,T],Y)}\right) \left(\tilde{H}_r + \alpha\right)^{-1}.$$

Let E be a Banach space and let $H \in E$. Let $\epsilon > 0$. We denote

$$(29) \quad \mathcal{B}_E(H, \epsilon) := \{\tilde{H} \in E : \|H - \tilde{H}\|_E < \epsilon\}.$$

Proposition 2. *Let $\epsilon \in (0, 1)$ and let $p \in (0, \frac{1}{2})$. Let $\alpha = 2\epsilon^p$ and let $\|H\|_{C([0,T],Y)} \leq M_2$. Let $H_r \in Y$ be positive semidefinite. Let us assume that $\tilde{H} \in \mathcal{B}_{C([0,T],Y)}(H, \epsilon)$. Then*

$$\left\|\mathbf{Z}_\alpha(H) - \mathbf{Z}_\alpha(\tilde{H})\right\|_{C([0,T],Y)} \leq 2^{-1}\epsilon^{1-2p}.$$

Proof. Using (28) we see that if $\Psi_\alpha\left(\left\|M_3 - (\tilde{H} + \alpha)\right\|_{C([0,T],Y)}\right) \neq 0$, then

$$\left\|M_3 - (\tilde{H} + \alpha)\right\|_{C([0,T],Y)} \leq M_3 - \frac{\alpha}{4} < M_3$$

and $\left(\tilde{H}_r + \alpha\right)^{-1}$ is defined by the formula

$$\left(\tilde{H}_r + \alpha\right)^{-1} = \frac{1}{M_3} \left(I - \frac{M_3 - (\tilde{H}_r + \alpha)}{M_3}\right)^{-1} = \frac{1}{M_3} \sum_{l=1}^{\infty} \left(\frac{M_3 - (\tilde{H}_r + \alpha)}{M_3}\right)^l.$$

This gives that $\mathbf{Z}_\alpha(\tilde{H})(r) \in Y$, when $r \in [0, T]$. Proposition 1 gives continuity for the map $r \mapsto \tilde{H}_r$. As $\left(\tilde{H}_r + \alpha\right) \mapsto \left(\tilde{H}_r + \alpha\right)^{-1}$ is continuous operation we see that $\mathbf{Z}_\alpha(\tilde{H}) \in C([0, T], Y)$. It remains to show

that the norm estimate holds. Having $\epsilon \in (0, 1)$, $\|H - \tilde{H}\|_{C([0, T], Y)} \leq \epsilon$, and $\alpha = 2\epsilon^p$ we have

$$\|M_3 - (\tilde{H} + \alpha)\|_{C([0, T], Y)} \leq M_3 - \frac{\alpha}{2}.$$

Thus $\Psi_\alpha^Z(\|M_3 - (\tilde{H} + \alpha)\|_{C([0, T], Y)}) = 1$ and $\mathbf{Z}_\alpha(\tilde{H})$ is the map

$$r \mapsto (\tilde{H}_r + \alpha)^{-1}.$$

Let $r \in [0, T]$. We denote

$$H_{\alpha, r} = (H_r + \alpha), \quad \tilde{H}_{\alpha, r} = (\tilde{H}_r + \alpha), \quad E = \tilde{H}_{\alpha, r} - H_{\alpha, r}.$$

As H_r is positive semidefinite we have

$$(30) \quad \|H_{\alpha, r}^{-1}\|_Y \leq \alpha^{-1}.$$

Moreover

$$\tilde{H}_{\alpha, r}^{-1} - H_{\alpha, r}^{-1} = ([I + H_{\alpha, r}^{-1}E]^{-1} - I)H_{\alpha, r}^{-1}.$$

Thus

$$(31) \quad \|(\tilde{H}_{\alpha, r})^{-1} - (H_{\alpha, r})^{-1}\|_Y \leq \frac{\|(H_{\alpha, r})^{-1}E\|_Y}{1 - \|(H_{\alpha, r})^{-1}E\|_Y} \|(H_{\alpha, r})^{-1}\|_Y.$$

We have $\frac{1}{2} \geq \frac{\epsilon}{\alpha}$. Using (30) and (31) we have

$$\|(\tilde{H}_{\alpha, r})^{-1} - (H_{\alpha, r})^{-1}\|_Y \leq \frac{\frac{\epsilon}{\alpha}}{1 - \frac{1}{2}} \|(H_{\alpha, r})^{-1}\|_Y \leq 2\frac{\epsilon}{\alpha^2} = 2^{-1}\epsilon^{1-2p}. \quad \square$$

Let P_r and B be as defined in (7). We define

$$(32) \quad \begin{aligned} \mathbf{S} : C([0, T], Y) &\rightarrow C([0, T]), \\ \mathbf{S}(\tilde{Z}_\alpha)(r) &= \langle \tilde{Z}_\alpha(r)P_r B1, B1 \rangle_{L^2(0, 2T)}. \end{aligned}$$

Proposition 3. *We have $\|\mathbf{S}\|_{C([0, T])} \leq \frac{T^3}{3}$.*

Proof. As the maps $r \mapsto P_r B1$ and $r \mapsto Z_\alpha(r)$ are continuous, we have that $\mathbf{S}(\tilde{Z}_\alpha) \in C([0, T])$. Let $r \in [0, T]$. We have

$$\|P_r\|_Y \leq 1, \quad \|B1\|_Y^2 = \frac{T^3}{3},$$

and therefore

$$|\mathbf{S}(\tilde{Z}_\alpha)(r)| = |\langle \tilde{Z}_\alpha(r)P_r B1, B1 \rangle_{L^2(0, 2T)}| \leq \frac{T^3}{3} \|\tilde{Z}_\alpha(r)\|_Y. \quad \square$$

Lemma 1. *Let $c \in \mathcal{D}(\mathcal{A})$. There is $C > 0$ such that for all $r > 0$ and $p \in H^1(M)$ satisfying $\text{supp}(p) \subset M(r)$ there is $f \in S_r$ such that $u^f(x, T) = p(x)$ and*

$$(33) \quad \|f\|_{L^2(0, 2T)} \leq C \|p\|_{H^1(M)}.$$

We recall that $M(r)$ is defined by (10) and S_r is defined by (11).

Proof. Let us consider the wave equation with time and space having the exchanged roles

$$(34) \quad \begin{aligned} (\partial_x^2 - c(x)^{-2} \partial_t^2) \tilde{u}(x, t) &= 0, & (x, t) &\in (0, \chi(T)) \times (0, T), \\ \tilde{u}(x, T) &= p(x), & x &\in [0, \chi(T)], \\ \tilde{u}(\chi(T), t) &= \partial_x \tilde{u}(\chi(T), t) = 0, & t &\in (0, T). \end{aligned}$$

By [37] the solution of (34) satisfies

$$(35) \quad \|\tilde{u}(0, \cdot)\|_{H^1(0, T)} \leq C \|p\|_{H^1(M(T))}.$$

If $\text{supp}(p) \subset M(r)$ then $\text{supp}(\tilde{u}(0, \cdot)) \subset [T - r, T]$ and $\tilde{u}(x, 0) = \partial_t \tilde{u}(x, 0) = 0$, when $x \in [0, \chi(T)]$, by finite speed of propagation. We choose $f(t) = \tilde{u}(0, t)$. \square

Let $f_{\alpha, r}$ be as in (12) and define

$$(36) \quad s_\alpha \in C([0, T]), \quad s_\alpha(r) := \langle f_{\alpha, r}, B1 \rangle_{L^2(0, 2T)}.$$

Lemma 2. *Let $\alpha \in (0, \min(1, \frac{1}{\chi(T)^2}))$. Let V be as defined in (18). Then there is $C > 0$, independent of α , such that*

$$\|s_\alpha - V\|_{C([0, T])} \leq C\alpha^{\frac{1}{4}}.$$

Proof. Let $r \in [0, T]$ and $\delta > 0$. Let us define $w_\delta \in H^1(M)$

$$w_\delta(x) = \begin{cases} 1, & \text{if } x \in (0, \chi(r)), \\ 1 - \frac{x - \chi(r)}{\delta}, & \text{if } x \in [\chi(r), \chi(r) + \delta], \\ 0, & \text{if } x \in (\chi(r) + \delta, \infty). \end{cases}$$

Using $c(x) > C_0$ we have

$$(37) \quad \|w_\delta - 1_{M(r)}\|_{L^2(M; dV)}^2 \leq \frac{\epsilon}{3C_0^2}.$$

When $\delta \in (0, \min(1, \frac{1}{\chi(T)}))$ we have

$$(38) \quad \|w_\delta\|_{H^1(M)}^2 \leq \chi(T) + \frac{\delta}{3} + \frac{1}{\delta} \leq \frac{3}{\delta}.$$

Below $C > 0$ denotes a constant that may grow between inequalities, and that depends only on m, C_0, C_1, L . Lemma 1 gives us f_δ for which $u^{f_\delta}(x, T) = w_\delta(x)$. Thus (38) implies

$$(39) \quad \|f_\delta\|_{L^2(0,2T)} \leq C \|w_\delta\|_{H^1(M)} \leq \frac{C}{\delta^{\frac{1}{2}}}.$$

Let $f \in S_r$. We define

$$(40) \quad G_{\alpha,r}(f) = \|u^f - 1_{M(r)}\|_{L^2(M;dV)}^2 + \alpha \|f\|_{L^2(0,2T)}^2.$$

Using (37) and (39) we have

$$(41) \quad G_{\alpha,r}(f_\delta) = \|w_\delta - 1_{M(r)}\|_{L^2(M;dV)}^2 + \alpha \|f_\delta\|_{L^2(0,2T)}^2 \leq \frac{\delta}{C} + \alpha \frac{C}{\delta}.$$

Functional (40) and the functional defined in Theorem 3 have the same minimizer $f_{\alpha,r}$. Using (9), (21), and (36) we have

$$\begin{aligned} \|s_\alpha - V\|_{C([0,T])}^2 &= \sup_{r \in [0,T]} |\langle f_{\alpha,r}, B1 \rangle_{L^2([0,2T])} - V(r)|^2 \\ &= \sup_{r \in [0,T]} |\langle u^{f_{\alpha,r}}(T), 1 \rangle_{L^2(M;dV)} - \langle 1_{M(r)}, 1 \rangle_{L^2(M;dV)}|^2 \\ &\leq C \sup_{r \in [0,T]} \|u^{f_{\alpha,r}}(T) - 1_{M(r)}\|_{L^2(M;dV)}^2 \leq C \sup_{r \in [0,T]} G_{\alpha,r}(f_{\alpha,r}) \\ &\leq C \sup_{r \in [0,T]} G_{\alpha,r}(f_\delta) \end{aligned}$$

Using (41) and choosing $\delta = \alpha^{\frac{1}{2}}$ we have

$$\|s_\alpha - V\|_{C([0,T])}^2 \leq C \alpha^{\frac{1}{2}}. \quad \square$$

Lemma 3. *Let V be as defined in (18). Let v be as defined in (19). If $c \in \mathcal{D}(\mathcal{A})$ then there exists $\tilde{m} > 0$ s.t.*

$$\|v\|_{C^2([0,T])} \leq \tilde{m} \quad \text{and} \quad \|V\|_{C^3([0,T])} \leq \tilde{m}.$$

Proof. Equations (17), (18), (19), and (20) with the chain rule and the formula for the derivatives of inverse functions give us the result. \square

For small $h > 0$ we consider the partition

$$(0, T) = (0, h) \cup [h, 2h) \cup [2h, 3h) \cup \dots \cup [Nh - h, Nh) \cup [Nh, T),$$

where $N \in \mathbb{N}$ satisfies $T - h \leq Nh < T$. We define a discretized and regularized approximation of the derivative operator ∂_r by

$$(42) \quad D_h : C([0, T]) \rightarrow L^\infty(0, T),$$

$$D_h(\tilde{s}_\alpha)(r) = \begin{cases} \frac{\tilde{s}_\alpha(h)}{h}, & \text{if } r \in (0, h), \\ \frac{\tilde{s}_\alpha(jh+h) - \tilde{s}_\alpha(jh)}{h}, & \text{if } r \in [jh, jh+h), \\ \frac{\tilde{s}_\alpha(T) - \tilde{s}_\alpha(Nh)}{h}, & \text{if } r \in [Nh, T). \end{cases}$$

Proposition 4. *Let $\beta > 0$ and $\epsilon \in (0, \min(\frac{1}{\beta^{\frac{1}{4}}}, \frac{1}{\beta^{\frac{1}{4}}\chi(T)^{\frac{1}{2}}}))$. Let $\alpha = \beta\epsilon^4$, $h = \epsilon^{\frac{1}{2}}$, V be as defined in (18) and let s_α be as defined in (36). Let us assume that $\tilde{s}_\alpha \in \mathcal{B}_{C([0,T])}(s_\alpha, \epsilon)$. Then*

$$\|D_h(\tilde{s}_\alpha) - \partial_r V\|_{L^\infty(0,T)} \leq C\epsilon^{\frac{1}{2}},$$

where C is independent of α and \tilde{s}_α .

Proof. Let $r \in [jh, jh+h)$. Using (42) we have

$$\begin{aligned} \left| D_h(\tilde{s}_\alpha)(r) - \partial_r V(r) \right| &= \left| \frac{\tilde{s}_\alpha(jh+h) - \tilde{s}_\alpha(jh)}{h} - \partial_r V(r) \right| \\ &\leq \left| \frac{\tilde{s}_\alpha(jh+h) - s_\alpha(jh+h)}{h} \right| + \left| \frac{s_\alpha(jh) - \tilde{s}_\alpha(jh)}{h} \right| \\ &\quad + \left| \frac{s_\alpha(jh+h) - V(jh+h)}{h} \right| + \left| \frac{V(jh) - s_\alpha(jh)}{h} \right| \\ &\quad + \left| \frac{V(jh+h) - V(jh)}{h} - \partial_r V(r) \right|. \end{aligned}$$

Lemma 3 gives us $\|V\|_{C^3([0,T])} \leq \tilde{m}$. When $r \in [jh, (jh+h)$ there is $\xi \in (jh, jh+h)$ such that

$$(43) \quad \left| \frac{V(jh+h) - V(jh)}{h} - \partial_r V(r) \right| = \left| \partial_r V(\xi) - \partial_r V(r) \right| \leq h\tilde{m}.$$

Using (43) and Lemma 2 with assumption we get

$$\left| D_h(\tilde{s}_\alpha)(r) - \partial_r V(r) \right| \leq \frac{2\epsilon}{h} + \frac{2C\alpha^{\frac{1}{4}}}{h} + h\tilde{m}.$$

Let us choose $h = \epsilon^{\frac{1}{2}}$ and $\alpha = \beta\epsilon^4$. Then

$$(44) \quad \left| D_h(\tilde{s}_\alpha)(r) - \partial_r V(r) \right| \leq C\epsilon^{\frac{1}{2}}.$$

The proof is almost identical when $r \in (0, h)$ or $r \in [Nh, T)$. Note that the right hand side of (44) is independent of r . □

Let C_0 and C_1 be as in (1). Let $\tilde{k}_\alpha \in L^\infty(0, T)$ and we define

$$\Psi^W(\tilde{k}_\alpha)(r) = \begin{cases} \frac{1}{C_1}, & \text{if } \tilde{k}_\alpha(r) < C_1^{-1}, \\ \frac{1}{\tilde{k}_\alpha(r)}, & \text{if } C_1^{-1} \leq \tilde{k}_\alpha(r) \leq C_0^{-1}, \\ \frac{1}{C_0}, & \text{if } \tilde{k}_\alpha(r) > C_0^{-1}. \end{cases}$$

We define

(45)

$$W : L^\infty(0, T) \rightarrow L^\infty(M), \quad W(\tilde{k}_\alpha)(r) = \begin{cases} \Psi^W(\tilde{k}_\alpha)(r), & \text{if } r \in (0, T), \\ 1, & \text{if } r \in [T, \infty). \end{cases}$$

Proposition 5. *Let V be as defined in (18) and v be as defined in (19). Let us assume that $\tilde{k}_\alpha \in \mathcal{B}_{L^\infty(0, T)}(\partial_r V, \epsilon)$. Then*

$$\|W(\tilde{k}_\alpha) - v\|_{L^\infty(M)} \leq C_1^2 \epsilon.$$

Proof. For all $x \in M$, we have $0 < C_0 \leq c(x) \leq C_1$. Let $r \in (0, T)$ and let assume that $C_1^{-1} \leq \tilde{k}_\alpha(r) \leq C_1^{-1}$. Using (19) and (20) we have $0 < \frac{1}{C_1} \leq \partial_r V(r) \leq \frac{1}{C_0}$. Then

$$(46) \quad \left| \frac{1}{\tilde{k}_\alpha(r)} - \frac{1}{\partial_r V(r)} \right| = \left| \frac{\tilde{k}_\alpha(r) - \partial_r V(r)}{\tilde{k}_\alpha(r) \partial_r V(r)} \right| \leq C_1^2 \epsilon.$$

In the case when $r \in (0, T)$ and $\tilde{k}_\alpha(r) < C_1^{-1}$ or $\tilde{k}_\alpha(r) > C_0^{-1}$ we obtain similar estimates. Note that the right hand side of (46) is independent of r . When $r \geq T$ the left hand side is identically zero. \square

Let $\tilde{w}_\alpha \in L^\infty(M)$ and we define

(47)

$$\Psi^\Phi : L^\infty(M) \rightarrow L^\infty(M), \quad \Psi^\Phi(\tilde{w}_\alpha)(r) := \begin{cases} C_0, & \text{if } \tilde{w}_\alpha(r) < C_0, \\ \tilde{w}_\alpha(r), & \text{if } C_0 \leq \tilde{w}_\alpha(r) \leq C_1, \\ C_1, & \text{if } \tilde{w}_\alpha(r) > C_1. \end{cases}$$

Let $\tilde{w}_\alpha \in L^\infty(M)$ and we define

$$(48) \quad \Upsilon : L^\infty(M) \rightarrow C(M), \quad \Upsilon(\tilde{w}_\alpha)(t) = \int_0^t \tilde{w}_\alpha(t') dt'.$$

Using (47) and (48) we see that $\Upsilon \circ \Psi^\Phi(\tilde{w}_\alpha) : M \rightarrow M$ is bijective as a function of t . Let us denote $\tilde{\chi} = \Upsilon \circ \Psi^\Phi(\tilde{w}_\alpha)$ and $\tilde{\chi}^{-1} = (\Upsilon \circ \Psi^\Phi(\tilde{w}_\alpha))^{-1}$.

We define the sixth operator by

(49)

$$\Phi : L^\infty(M) \rightarrow L^\infty(M), \quad \Phi(\tilde{w}_\alpha) = \begin{cases} \tilde{w}_\alpha \circ \tilde{\chi}^{-1}, & \text{if } x \in [0, L), \\ 1, & \text{if } x \in [L, \infty). \end{cases}$$

Proposition 6. *Let $\epsilon > 0$ and v be as defined in (19). Let us assume that $\tilde{w}_\alpha \in \mathcal{B}_{L^\infty(M)}(v, \epsilon)$. Then*

$$\|\Phi(\tilde{w}_\alpha) - c\|_{L^\infty(M)} \leq C\epsilon.$$

Proof. Let us denote $t = \chi^{-1}(x)$ and $\tilde{t} = \tilde{\chi}^{-1}(x)$. Let $x \in [0, L)$. Using (20) and (49) we have

$$|\Phi(\tilde{w}_\alpha)(x) - c(x)| = |\tilde{w}_\alpha(\tilde{t}) - v(t)| \leq |\tilde{w}_\alpha(\tilde{t}) - v(\tilde{t})| + |v(\tilde{t}) - v(t)|.$$

Lemma 3 gives us $\|v\|_{C^2(0,T)} \leq \tilde{m}$ and we have

$$(50) \quad |v(\tilde{t}) - v(t)| \leq \tilde{m}|\tilde{t} - t|.$$

Using (1) and (20) we see that $0 < C_0 \leq v(t) \leq C_1$ and we have

$$(51) \quad C_0|\tilde{t} - t| \leq \left| \int_t^{\tilde{t}} v(t') dt' \right| = |\chi(\tilde{t}) - \chi(t)|.$$

Having $\tilde{\chi}(\tilde{t}) = x = \chi(t)$ and using (20) and (48) we see that

$$(52) \quad |\chi(\tilde{t}) - \chi(t)| = |\chi(\tilde{t}) - \tilde{\chi}(\tilde{t})| = \left| \int_0^{\tilde{t}} (v(t') - \tilde{w}_\alpha(t')) dt' \right| \leq \tilde{\chi}^{-1}(L)\epsilon.$$

Using (50), (51), and (52) we have

$$(53) \quad |\Phi(\tilde{w}_\alpha)(x) - c(x)| \leq \left(1 + \frac{\tilde{m}\tilde{\chi}^{-1}(L)}{C_0} \right) \epsilon.$$

Note that the right hand side in (53) does not depend on x . When $x \in [T, \infty)$ the left hand side is identically zero. \square

Proof of Theorem 2. Let

$$(54) \quad \epsilon_0 = \min\left\{1, \frac{1}{2T}, \frac{1}{2^{\frac{13}{4}}T^9}, \frac{1}{2^{\frac{13}{4}}T^9\chi(T)^{\frac{9}{2}}}\right\}.$$

Suppose that $\tilde{\Lambda} \in \mathcal{B}_Y(\Lambda, \epsilon)$. We denote $H = \mathbf{H}\Lambda$ and $\tilde{H} = \mathbf{H}\tilde{\Lambda}$. Using Proposition 1 we get

$$\|H - \tilde{H}\|_{C([0,T],Y)} \leq 2T\epsilon.$$

We denote $Z_\alpha = \mathbf{Z}_\alpha(H)$ and $\tilde{Z}_\alpha = \mathbf{Z}_\alpha(\tilde{H})$. We have $\tilde{H} \in \mathcal{B}_{C([0,T],Y)}(H, 2T\epsilon)$ and $\epsilon \in (0, \min(1, \frac{1}{2T}))$. Proposition 2 with $p = \frac{4}{9}$ gives us

$$\|Z_\alpha - \tilde{Z}_\alpha\|_{C([0,T],Y)} \leq 2^{-2p}T^{1-2p}\epsilon^{1-2p} =: \epsilon_1,$$

since $\alpha = 2^{p+1}T^p\epsilon^p = 2^{\frac{13}{9}}T^{\frac{4}{9}}\epsilon^{\frac{4}{9}}$.

We denote $s_\alpha = \mathbf{S}Z_\alpha$ and $\tilde{s}_\alpha = \mathbf{S}\tilde{Z}_\alpha$. We have $\tilde{Z}_\alpha \in \mathcal{B}_{C([0,T],Y)}(Z_\alpha, \epsilon_1)$. Proposition 3 gives us

$$\|s_\alpha - \tilde{s}_\alpha\|_{C([0,T])} \leq \frac{2^{-2p}}{3}T^{4-2p}\epsilon^{1-2p} =: \epsilon_2.$$

We denote $\tilde{k}_\alpha = D_h(\tilde{s}_\alpha)$. We have $\tilde{s}_\alpha \in \mathcal{B}_{C([0,T])}(s_\alpha, \epsilon_2)$ and $\epsilon_2 \in \left(0, \min\left(\frac{1}{\beta^{\frac{1}{4}}}, \frac{1}{\beta^{\frac{1}{4}}\chi(T)^{\frac{1}{2}}}\right)\right)$. Proposition 4 with $\beta = 3^4 2^5 T^{-\frac{76}{9}}$ gives us

$$\left\| \tilde{k}_\alpha - \partial_r V \right\|_{L^\infty(0,T)} \leq C \frac{2^{-p}}{3^{\frac{1}{2}}} T^{2-p} \epsilon^{\frac{1}{2}-p} =: \epsilon_3,$$

since $\alpha = \beta^{\frac{2-8p}{34}} T^{16-8p} \epsilon^{4-8p} = 2^{\frac{13}{9}} T^{\frac{4}{9}} \epsilon^{\frac{4}{9}}$.

We denote $\tilde{w}_\alpha = W(\tilde{k}_\alpha)$. We have $\tilde{k}_\alpha \in \mathcal{B}_{L^\infty(0,T)}(\partial_r V, \epsilon_3)$. Proposition 5 gives us

$$\left\| \tilde{w}_\alpha - v \right\|_{L^\infty(M)} \leq C_1^2 C \frac{2^{-p}}{3^{\frac{1}{2}}} T^{2-p} \epsilon^{\frac{1}{2}-p} =: \epsilon_4.$$

We denote $\tilde{c}_\alpha = \Phi(\tilde{w}_\alpha)$. We have $\tilde{w}_\alpha \in \mathcal{B}_{L^\infty(M)}(v, \epsilon_4)$. Proposition 6 gives us

$$\left\| \tilde{c}_\alpha - c \right\|_{L^\infty(M)} \leq C C_1^2 \frac{2^{-p}}{3^{\frac{1}{2}}} T^{2-p} \epsilon^{\frac{1}{2}-p}.$$

Let $\epsilon \in (0, \epsilon_0)$. Using (22), (28), (32), (42), (45), and (49) we define

$$(55) \quad \begin{aligned} \mathcal{R}_{\alpha(\epsilon)} : Y &\rightarrow X, \\ \mathcal{R}_{\alpha(\epsilon)} &= \Phi \circ W \circ D_h \circ \mathbf{S} \circ \mathbf{Z}_\alpha \circ \mathbf{H}, \end{aligned}$$

and we have an estimate

$$\left\| \mathcal{R}_{\alpha(\epsilon)}(\tilde{\Lambda}) - c \right\|_X \leq C \epsilon^{\frac{1}{18}}.$$

□

APPENDIX A: THE DIRECT PROBLEM

For the convenience of reader, we give the proof of the following, quite well known result, for continuity of the direct problem in our setting.

Theorem 4. *Let $c \in \mathcal{D}(\mathcal{A})$ and $f \in L^2(0, 2T)$. Then the boundary value problem (3) has a unique solution $u^f \in H^1((0, 2T) \times M)$. The operators Λ and U_T , defined in (4) and (8), are linear and bounded operators, and the direct map \mathcal{A} , defined in (5), is continuous.*

Proof. Let consider the wave equation (3). When $c = 1$ on M we denote the solution by u_0^f and have

$$u_0^f(t, x) = h(t - x), \quad h(s) = \begin{cases} -\int_0^s f(t) dt, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Notice that $f \mapsto u_0^f$ is continuous from $L^2(0, 2T)$ to $H^1((0, 2T) \times M)$. Let us now show that the same is true for $f \mapsto u^f$.

We choose $\psi \in C^\infty(M)$ such that $\psi = 1$ near $x = 0$ and $c = 1$ in the support of ψ . The commutator $A = [\partial_x^2, \psi]$ is a first order differential operator, whence $Au_0^f \in L^2((0, 2T) \times M)$ for $f \in L^2(0, 2T)$. Let w be the solution of

$$\begin{aligned} (\partial_t^2 - c(x)^2 \partial_x^2)w(t, x) &= Au_0^f \quad \text{in } (0, 2T) \times M, \\ \partial_x w(t, 0) &= 0, \\ w|_{t=0} = \partial_t w|_{t=0} &= 0. \end{aligned}$$

Then $w \in H^1((0, 2T) \times M)$, see e.g. [36], and $u^f := \psi u_0^f - w \in H^1((0, 2T) \times M)$ is the solution of (3). Indeed, $c^2 \partial_x^2 \psi = \psi \partial_x^2 + A$, where ψ is interpreted as a multiplication operator. Here we are using the fact that $c = 1$ in the support of ψ . Thus

$$(\partial_t^2 - c^2 \partial_x^2)u^f = \psi(\partial_t^2 - \partial_x^2)u_0^f + Au_0^f - Au_0^f = 0.$$

As $\psi = 1$ near $x = 0$, we see that u satisfies also the boundary conditions in (3).

Let us now suppose that $f \in C_0^\infty(0, T)$. Let u^f be solution for the boundary value problem in (3). Let c be as defined in (1). Let $x \in M$ and we define

$$(56) \quad k \in C^2(M), \quad k(x) = \exp\left(\frac{1}{2c(x)}\right)$$

and

$$(57) \quad G : C^2(M \times (0, 2T)) \rightarrow C(M \times (0, 2T)),$$

$$G(u) = k^{-1} \left(\partial_t^2 - c^2 \partial_x^2 \right) k u = \left(\partial_t^2 - c^2 \partial_x^2 - 2c^2 k^{-1} \partial_x k \partial_x - c^2 k^{-1} \partial_x^2 k \right) u.$$

Let $x \in M$ and define

$$(58) \quad \phi(x) = \int_0^x c(x')^{-1} dx'.$$

Let us denote $\tilde{x} = \phi(x)$ and define

$$(59) \quad \tilde{u}^f(\tilde{x}, t) = \tilde{u}^f(\phi(x), t) := \frac{u^f(x, t)}{k(x)}.$$

Using (56), (57), (58), and (59) we see that $\tilde{u}^f(\tilde{x}, t)$ is a solution of the boundary value problem

$$\begin{aligned} (60) \quad (\partial_t^2 - \partial_{\tilde{x}}^2 + q(\tilde{x}))\tilde{u}^f(\tilde{x}, t) &= 0, & (\tilde{x}, t) &\in M \times (0, 2T), \\ \partial_{\tilde{x}} \tilde{u}^f(0, t) &= e^{\frac{1}{2}} f(t), & \partial_x \tilde{u}^f(L, t) &= 0, & t &\in (0, 2T), \\ \tilde{u}^f(x, 0) = \partial_t \tilde{u}^f(x, 0) &= 0, & \tilde{x} &\in M, \end{aligned}$$

where

$$(61) \quad q(\tilde{x}) = -c^2(\phi^{-1}(\tilde{x}))k^{-1}(\phi^{-1}(\tilde{x}))\partial_x^2 k(\phi^{-1}(\tilde{x})).$$

We define $\lambda_q f = \tilde{u}|_{\tilde{x}=0}$. Let us consider two velocity functions c_1 and c_2 , and let q_1 and q_2 be the potentials corresponding to c_1 and c_2 via formula (61). Using (59) we have

$$(62) \quad \|\Lambda_{c_1} - \Lambda_{c_2}\|_{\mathcal{L}(L^2(0,2T))} \leq e^{\frac{1}{2}} \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(L^2(0,2T))}.$$

Let us denote by $u_{q_1}^f$ and $u_{q_2}^f$ the two solutions with respect to potentials q_1 and q_2 for the problem (60). Let us define $w(\tilde{x}, t) = \tilde{u}_{q_1}^f(\tilde{x}, t) - \tilde{u}_{q_2}^f(\tilde{x}, t)$. Then w is the solution of

$$(63) \quad \begin{aligned} (\partial_t^2 - \partial_x^2 + q_1(\tilde{x}))w(\tilde{x}, t) &= F(\tilde{x}, t), & (\tilde{x}, t) &\in M \times (0, 2T), \\ \partial_{\tilde{x}} w(0, t) &= 0, \quad \partial_{\tilde{x}} w(L, t) = 0, & t &\in (0, 2T), \\ w(\tilde{x}, 0) &= \partial_t w(\tilde{x}, 0) = 0, & x &\in M, \end{aligned}$$

where

$$(64) \quad F(\tilde{x}, t) = (q_1(\tilde{x}) - q_2(\tilde{x}))\tilde{u}_{q_2}^f(\tilde{x}, t).$$

By [37] we have an estimate

$$(65) \quad \|w\|_{H^1(M \times (0, 2T))} \leq C \|F\|_{L^2(M \times (0, 2T))}.$$

Let $\hat{T} = \phi(L)$ and let $\text{supp}(u_{q_2}^f) \subset [0, \hat{T}]$. Using (64) we have

$$(66) \quad \|F\|_{L^2(M \times (0, 2T))} \leq \|q_1 - q_2\|_{L^\infty(0, \hat{T})} \|u_{q_2}^f\|_{H^1(M \times (0, 2T))}.$$

Because $f \mapsto \tilde{u}_{q_2}^f$ is continuous from $L^2(0, 2T)$ to $H^1(M \times (0, 2T))$ we have

$$(67) \quad \|\tilde{u}_{q_2}^f\|_{H^1(M \times (0, 2T))} \leq C \|f\|_{L^2(0, 2T)}.$$

Using (4) we see that

$$(68) \quad \|\Lambda_{q_1} f - \Lambda_{q_2} f\|_{L^2(0, 2T)} \leq C \|u_{q_1}^f - u_{q_2}^f\|_{H^1(M \times (0, 2T))}.$$

Using (66), (65), (67), and (68) we have

$$(69) \quad \|\Lambda_{q_1} - \Lambda_{q_2}\|_{\mathcal{L}(L^2(0, 2T))} \leq C \|q_1 - q_2\|_{L^\infty(0, T)}.$$

Using (61) we have $q(\tilde{x}) = -c^2(x)k^{-1}(x)\partial_x^2 k(x)|_{x=\phi^{-1}(\tilde{x})}$. Let $x \in M$ and define

$$h_1(x) = -c_1^2(x)k_1^{-1}(x)\partial_x^2 k_1(x), \quad h_2(x) = -c_2^2(x)k_2^{-1}(x)\partial_x^2 k_2(x).$$

We have

$$\begin{aligned}
|h_1(x) - h_2(x)| &= |c_2^2(x)k_2^{-1}(x)\partial_x^2 k_2(x) - c_1^2(x)k_1^{-1}(x)\partial_x^2 k_1(x)| \\
&\leq |c_1^2(x) - c_2^2(x)||k_1^{-1}(x)||\partial_x^2 k_1(x)| \\
&\quad + |k_1^{-1}(x) - k_2^{-1}(x)||c_2^2(x)||\partial_x^2 k_1(x)| \\
&\quad + |\partial_x^2 k_1(x) - \partial_x^2 k_2(x)||c_2^2(x)||k_2^{-1}(x)|
\end{aligned}$$

Using (56) we can bound each of these three terms and have

$$(70) \quad \|q_1 - q_2\|_{L^\infty(0,\hat{T})} \leq C \|c_1 - c_2\|_{C^2(0,L)}.$$

Now (62), (68), and (70) imply that \mathcal{A} is continuous. \square

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